For this problem set, you may find it useful to consult Ken Rosen’s textbook *Discrete Math and Its Applications*.

1. Give the contrapositive of the following statement. “If every bird flies, then there is a hungry cat.”

Answer:

* If there is not a hungry cat, then not every bird flies.

1. A proposition is a statement that can be true or false but not both. Let A, B, and C be propositions. Let denote logical AND, let denote logical OR, and let denote logical NOT. Argue that if is true, then must be true as well.

Answer:

* is true if all possible values in its domain map to “True” in its range.
* The table below maps each possibility of A, B and C through each of the terms of the expression
* In each case the resulting value is true
* Therefore , or if is true, then must be true as well.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| A | B |  | C |  |  |  |  |  |
| T | T | F | T | T | T | T | T | T |
| T | T | F | F | T | F | F | T | T |
| T | F | T | T | T | T | T | T | T |
| T | F | T | F | T | T | T | T | T |
| F | T | F | T | T | T | T | T | T |
| F | T | F | F | T | F | F | F | T |
| F | F | T | T | F | T | F | T | T |
| F | F | T | F | F | T | F | F | T |

1. We use the notation to indicate that A implies B. This new proposition is true except when A is true and B is false. We write when either both A and B are true or both are false. Argue that if and only if and .

Answer:

* Two expressions are equivalent if they map to the same values in a range for any values in their domain
* All possible values of the domain are given in the truth table
* If the values in the range of each expression match for every possible combination of values in the domain, they are equivalent
* has the truth table shown below, demonstrating that is true except when A is true and B is false.
* Similarly is true except when B is true and A is false (see table)
* is defined to be true when either both A and B are true or both are false (see table).
* and can be represented as and is also true only when either both A and B are true or both are false (see table).
* Hence if and only if and .

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| A | B |  |  |  |  |
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

1. We will use the notation to indicate the number of elements in the set or its *cardinality*, e.g. is the number of elements in the set A. Consider four sets A,B,C,D such that the intersection of any three is empty. Use the inclusion-exclusion to give an expression for without using any union ( symbols.

Answer:

* Since the intersection of any three sets is empty, all terms with greater than three elements may be dropped as empty

1. State the formal definition of , and show that the function is .

Answer:

Definition: Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that is if there are constants C and k such that

whenever . [Source: Rosen, *Discrete Mathematics and Its Applications, 7e*, pp. 205]

* Assuming , we can drop the absolute values to find a C such that:
* First:
* and implies that:

, , and

* Choose and “increase” the elements of to simplify:
* Thus the function is because , whenever

1. Let A be a set. We use the notation to indicate the power set of A, which consist of all subsets of A. For example, if , then . Consider and use an inductive argument to show that

for all positive integers n.

Answer:

* The following proof is by induction.
* **Basis**: First demonstrate P(1) = 1:
* **Induction step**: Then assume and show that it is true for:
* Thus the formula is correct for which proves the result.

1. Prove that the set of all languages over that have a bounded maximum string length is countable.

Answer:

* Let
* Define language L,
* Let be the set of all strings of length
* Any of these is countable because there exists a bijection of the positive integers onto the elements of the set. There are, however, infinitely many of these sets.
* To address this we use the following procedure: By putting each  in size order across the top of a grid, and successive natural numbers down the left side of the grid, we view a mapping/bijection from 1, to , 2 to , and so on: anti-diagonal mappings starting at each element in the left of the array, going up-right.
* Each diagonal set has a finite number of elements (and is thus countable), so each element in each set will be covered by the anti-diagonalization.
* Thus, there will exist a bijection of positive integers onto these diagonals (and in turn their elements), meaning that the countable (yet not finite) number of countable sets is also countable as a whole.
* Therefore the set of all languages over that have a bounded maximum string length is countable